Eigen values and eigen vectors

SHORT ANSWERS:

1. Define characterstic equation?

sol~ Let 'A' be a $n \times n$ matrix. Let 'x' be a eigen vector of 'A' corresponding to the eigen value 'λ' then $AX = \lambda X$

$$AX - \lambda X = 0$$
$$= (A - I\lambda)X = 0$$

Where $A - \lambda I$ is characteristic matrix of 'A' also $|A - \lambda I| = 0$ is called characteristic equation of 'A'

2. Find the characteristic roots of matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

sol~ Given matrix is $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

ie, $\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)[(3-\lambda)(2-\lambda)-2] - 2[2-\lambda-1] + 1[2-3+\lambda] = 0$

$\Rightarrow \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$, on simplification

$\Rightarrow (\lambda-1)(\lambda^2 - 6\lambda + 5) = 0$

$\Rightarrow (\lambda-1)(\lambda-1)(\lambda-5) = 0$

$\therefore \lambda = 1, 1, 5$

$\therefore$ The characteristic roots of A are 1, 1, 5

3. Find the sum and product of eigen values of matrix

$$\begin{bmatrix} 2 & 1 & -1 \\ 3 & 4 & 2 \\ 1 & 0 & 2 \end{bmatrix}$$

Sol:- Given, matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 4 & 2 \\ 1 & 0 & 2 \end{bmatrix}$

Sum of eigenvalues = trace of the matrix

= Sum of the diagonal elements

$= 2+4+2 = 8$

product of eigenvalues $= \det A = \begin{vmatrix} 2 & 1 & -1 \\ 3 & 4 & 2 \\ 1 & 0 & 2 \end{vmatrix}$

$= 2(8-0) - 1(6-2) - 1(0-4) = 16 - 4 + 4 = 16$

4. Define Hermitian and skew Hermitian

A square Matrix A such $A^T = \bar{A}$ or $(\bar{A})^T = A$ is called Hermitian matrix

A squar Matrix A such that $\bar{A} = -\bar{A}$ or $(\bar{A})^T = A$ is called skew-Hermitian Matrix.

5. Using cayley-Hamilton theorem, find A^8 if $A=\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

Sol:- Given $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

characteristic equation of A is $|A-\lambda I|=0$

i.e $\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix}=0$

$\Rightarrow \lambda^2-5=0$

By cayley-Hamilton theorem, A satisfies its chara-cteristic equation. so we must have $A^2=5I$.

$\therefore A^8 = 5A^6 = 5(A^2)(A^2)(A^2)$

$= 5(5I)(5I)(5I)$

$= 625I$

6. Define Index, Signature and Nature.

Index:- The number of positive terms in canonical form or normal form of a quadratic form is known as the index of the quadratic form.

It is denoted by 'S'.

Signature:- The signature of quadratic form, is the excess no. of '+ve terms over the negative terms i.e Signature = [(no. of +ve terms) − (no. of −ve terms)]

Note:- if 'r' is the rank and 'S' is the index of quadratic form then signature = 2S − r

$$\boxed{\text{Signature} = 2S - r}$$

Nature of quadratic form:-

The quadratic form $X^T A X$ in n variables is said to be

i) positive definite:- if $r = n$ and $s = n$ (or) if all the eigen values of A are positive.

ii) Negative definite:- if $r = n$ and $s = 0$ (or) if all the eigen values of A are negative

iii. positive semidefinite:- if $r < n$ and $s = r$ (or) if all the eigen values of $A \geq 0$ and at least one eigen value is zero

iv. Negative semidefinite:- if $r < n$ and $s = 0$ (or) if all the eigen values of $A \leq 0$ and atleast one eigen value is zero

v. indefinite:- in all other cases

Note:- if the quadratic form Q is negative definite then −Q is positive definite.

7 Find the nature of the Quadratic form

$$2x^2+2y^2+2z^2+2yz$$

Sol: Given quadratic form is $2x^2+2y^2+2z^2+2yz$

Matrix of the quadratic form is $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Characteristic equation of A is $\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)[(2-\lambda)^2-1]=0$

$\Rightarrow (2-\lambda)(\lambda^2-4\lambda+3)=0$

$\Rightarrow (2-\lambda)(\lambda-3)(\lambda-1)=0$

$\therefore$ The roots of the characteristic equation are 1,2,3

All the roots are positive. The Q.F is +ve definite.

8 Reduce the Quadratic form to matrix form

$$x^2+4xy+6xz-y^2+2yz+4z^2$$

Sol: Given quadratic form is

$$x^2+4xy+6xz-y^2+2yz+4z^2$$

This can be written in matrix form as $[x\ y\ z]A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{bmatrix}$

characteristic equation of A is $|A-\lambda I|=0$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 1 \\ 3 & 1 & 4-\lambda \end{vmatrix}=0$$

$\Rightarrow (1-\lambda)\left[(-1-\lambda)(4-\lambda)-1\right]-2(8-2\lambda-3)+3(2+3+3\lambda)=0$

$\Rightarrow \lambda^3-4\lambda^2-15\lambda=0$

$\Rightarrow \lambda(\lambda^2-4\lambda-15)=0$

$\Rightarrow \lambda=0 \text{ (or) } \lambda^2-4\lambda-15=0$

$\lambda=0,\ 2+\sqrt{19},\ 2-\sqrt{19}$

Thus the quadratic form is indefinite.

9 Reduce matrix form to quadratic form $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Given

matrix $A=\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

An expression from matrix form to quadratic form

$Q=X^T A X$

$X=\begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad X^T=\begin{bmatrix} a & b & c \end{bmatrix}$

ⓐ (4)

Now

$$Q = [a\ b\ c]\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$[a\ b\ c]\begin{bmatrix} 6a & -2b & 2c \\ -2a & 3b & -c \\ 2a & -b & 3c \end{bmatrix}$$

$$\Rightarrow a(6a-2b+2c) + b(-2a+3b-c] + c(2a-b+3c)$$

$$\Rightarrow 6a^2 - 2ab + 2ac - 2ab + 3b^2 - bc + 2ac - bc + 3c^2$$

$$= 6a^2 + 3b^2 + 3c^2 - 4ab + 4ac - 2bc.$$

10. State cayley's Hamilton Theorem.

Ans Every Square Matrix satisfies its own characteristic equation.

EASSY ANSWERS :

1) Find the Eigen values and Eigen Vectors of the matrix

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Sol:- Given,

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Charasteristic equation Of matrix A is $|A-\lambda I|=0$

$$|A-\lambda I|=0$$

$$\left|\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right| = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)\left[(7-\lambda)(3-\lambda)-16\right]-(-6)\left[-6(3-\lambda)+8\right]+2\left[24-2(7-\lambda)\right]=0$$

$$(8-\lambda)\left[21+\lambda^2-7\lambda-3\lambda-16\right]+6\left[-18+6\lambda+8\right]+2\left[24-14+2\lambda\right]=0$$

$$(8-\lambda)\left[\lambda^2-10\lambda+5\right]+6\left[6\lambda-10\right]+2\left[2\lambda+10\right]=0$$

$$8\lambda^2-80\lambda+40-\lambda^3+10\lambda^2-5\lambda+36\lambda-60+4\lambda+20=0$$

$$-\lambda^3+18\lambda^2-45\lambda=0$$

$$\lambda=15,3,0$$

Eigen values are 15, 3, 0

To find Eigen vectors to corresponding Eigen values of

$$X[A-\lambda I]=0$$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$$

Eigen vector at the Eigen value $(\lambda)=0$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$$

$R_2 \rightarrow 4R_2+3R_1$

$R_3 \rightarrow 4R_3-R_1$

$$\begin{bmatrix} 8 & -6 & 2 \\ 0 & 10 & -10 \\ 0 & -10 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$$

$R_3 \rightarrow R_3+R_2$

$$\begin{bmatrix} 8 & -6 & 2 \\ 0 & 10 & -10 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$$

Rank(r) = 2

No. of unknowns (n) = 3

$$n-r=3-2$$
$$=1$$

perameter = 1

$8x-6y+2z=0 \longrightarrow (1)$

$10y-10z=0 \longrightarrow (2)$

let $z=k$.

$z = k$ sub in Eqn ②

$10y - 10k = 0$

$y = k$

Substitute y, z values in Eqn ①

$8x - 6k + 2k = 0$

$8x = 4k$

$x = k/2$

Eigen vector of Eigen value $\lambda = 0$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k/2 \\ k \\ k \end{bmatrix}$

$= 2k \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Eigen vector at Eigen value$(\lambda) = 3$

$$\begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_2 \to 5R_2 + 6R_1$

$R_3 \to 5R_3 - 2R_1$

$$\begin{bmatrix} 5 & -6 & 2 \\ 0 & -16 & -8 \\ 0 & -8 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_3 \to 2R_3 - R_2$

$$\begin{bmatrix} 5 & -6 & 2 \\ 0 & -16 & -8 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Rank (r) = 2

$n = 3$

perameter = 3-2

= 1

$5x - 6y + 2z = 0 \longrightarrow (3)$

$-16y - 8z = 0 \longrightarrow (4)$

let $z = k$. sub in (4)

$-16y - 8k = 0$

$y = -k/2$

Substitute y, z in Eqⁿ (3)

$5x - 6(-k/2) + 2(k) = 0$

$5x + 3k + 2k = 0$

$5x = -5k$

$x = -k$

Eigen vector of Eigen value $\lambda = 3$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ -k/2 \\ k \end{bmatrix}$

$= 2k \begin{bmatrix} -2 \\ -1 \\ -2 \end{bmatrix}$

Eigen vector at $\lambda = 15$

$$\begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_2 \longrightarrow 7R_2 - 6R_1$

$R_3 \longrightarrow 7R_3 + 2R_1$

$$\begin{bmatrix} -7 & -6 & 2 \\ 0 & -20 & -40 \\ 0 & -40 & -80 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_3 \longrightarrow R_3 - 2R_2$

$$\begin{bmatrix} -7 & -6 & 2 \\ 0 & -20 & -40 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\text{Rank}(r) = 2$, $n = 3$

$p = n - r$
$= 3 - 2$
$= 1$

$-7x - 6y + 2z = 0 \longrightarrow (5)$

$-20y - 40z = 0 \longrightarrow (6)$

let $z = k$. sub in Eq^n (6)

$-20y - 40k = 0$

$y = -2k$

Substitute y, z in Eq^n (5)

$-7x - 6(-2k) + 2(k) = 0$

$-7x + 12k + 2k = 0$

$x = 2k$.

Eigen vector at $\lambda = 15$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2k \\ -2k \\ k \end{bmatrix}$

$$= k \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

(2) Find the Eigen values and corresponding Eigen vectors of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Sol:- Given,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Charasteristic equation of matrix A is $|A - \lambda I| = 0$

$$\left| \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(1-\lambda)^2 - 1\right] - 1(1-\lambda-1) + 1(1-1+\lambda) = 0$$

$$(1-\lambda)^3 - (1-\lambda) + \lambda + \lambda = 0$$

$$-\lambda^3 + 3\lambda^2 = 0$$

$$\lambda^2(-\lambda + 3) = 0$$

$$\lambda = 0, 0, 3$$

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Eigen values are 0,0,3

To find Eigen vector of corresponding Eigen values is

$$[A-\lambda I]X=0$$

$$\begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$$

Eigen vector at $\lambda=0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$$

$R_2 \rightarrow R_2 - R_1$

$R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$$

Rank = 1 , n = 3

$p = n - r$
$= 3 - 1$
$= 2$

$x+y+z=0 \longrightarrow$ ①

let $y=k_1$, $z=k_2$ sub in Eqⁿ ①

$x+k_1+k_2=0$

$x=-(k_1+k_2)$

Eigen vector at $\lambda=0$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k_1-k_2 \\ k_1 \\ k_2 \end{bmatrix}$

$$= k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Eigen vector a $\lambda = 3$.

$$\begin{bmatrix} 1-3 & 1 & 1 \\ 1 & 1-3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_2 \to 2R_2 + R_1$

$R_3 \to 2R_2 + R_1$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Rank $= 2$, $n = 3$

$p = n - \cancel{r}$

$= 3 - 2$

$= 1$

$\Rightarrow 2x + y + z = 0 \longrightarrow (2)$

$-3y + 3z = 0 \longrightarrow (3)$

let $z = k$ sub in (3)

$-3y + 3k = 0$

$y = k,$

Substitute y, z in Eq^n (2)

$-2x + k + \cancel{k} = 0$

$-2x + k + k = 0$

$x = k.$

Eigen vector at $\lambda = 3$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix}$

$= k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

5) Determine model matrix of $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$

also find a) A^8.

b) A^4.

Sol:- Characteristic matrix of $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$ is

$|A - \lambda I| = 0$

$$\left| \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{vmatrix} = 0$$

$(1-\lambda)\left[(2-\lambda)(3-\lambda)-4\right]-1\left[4\right]+1\left[+4(2-\lambda)\right]=0$

$(1-\lambda)\left[6-5\lambda+\lambda^2-4\right]-4+(8-4\lambda)=0$

$(1-\lambda)\left[\lambda^2-5\lambda+2\right]+4-4\lambda=0$

$\lambda^2-5\lambda+2-\lambda^3+5\lambda^2-2\lambda+4-4\lambda=0$

$-\lambda^3+6\lambda^2-11\lambda+6=0$

$\lambda=1,2,3$

Eigen values are 1, 2, 3.

To find Eigen vectors of corresponding Eigen values is

$|A-\lambda I|X=0$

$$\begin{bmatrix}1-\lambda & 1 & 1\\ 0 & 2-\lambda & 1\\ -4 & 4 & 3-\lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=0$$

Eigen vector at $\lambda=1$

$$\begin{bmatrix}0 & 1 & 1\\ 0 & 1 & 1\\ -4 & 4 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=0$$

$R_3 \longrightarrow R_3 \Leftrightarrow R_1$

$$\begin{bmatrix}-4 & +4 & 2\\ 0 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=0$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix}-4 & +4 & 2\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=0$$

Rank(r) = 2 , n = 3

$p = 3-2$
$= 1$

$-4x - 4y + 2z = 0 \longrightarrow (1)$

$y + z = 0 \longrightarrow (2)$

let $z = k$ sub in (2)

$y = -k.$

Substitute y, z in (1)

$-4x + 4(-k) + 2k = 0$

$-4x - 4k + 2k = 0$

$x = 2k/4$

$x = k/2 = -k/2$

Eigen vector's is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k/2 \\ -k \\ k \end{bmatrix}$

$= 2k \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$

Eigen vector at $\lambda = 2$

$$\begin{bmatrix} 1-2 & 1 & 1 \\ 0 & 2-2 & 1 \\ -4 & 4 & 3-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_3 \to R_3 + 3R_2$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$r = 2, \ n = 3$

$P = 3 - 2$

$= 1$

$-x + y + z = 0 \longrightarrow (3)$

$z = 0$

let $y = k$ sub in (3)

$-x + k + 0 = 0$

$x = k$

Eigen vector at $\lambda = 2$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix}$

$= k \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Eigen vector at $\lambda = 3$

$$\begin{bmatrix} 1-3 & 1 & 1 \\ 0 & 2-3 & 1 \\ -4 & 4 & 3-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ -4 & 4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$r = 2 \quad , n = 3$

$p = 3 - 2$

$= 1$

$-2x + y + z = 0 \longrightarrow (4)$

$-y + z = 0 \longrightarrow (5)$

let $z = k$ sub in (5)

$-y + k = 0$

$y = k.$

substitute y, z in (4)

$-2x + k + k = 0$

$-2x = -2k$

$x = k$

Eigen vector at $\lambda = 3$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix}$

$= k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Modal matrix $P = [x \quad y \quad z]$

$$P = \begin{bmatrix} -1 & 1 & 1 \\ -2 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

Diagnolization is $P^{-1}AP = D$

$$P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 2 & 1 \\ 4 & -3 & -1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 2 & 1 \\ 4 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & +4 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ -2 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

To find A^8

$\therefore \boxed{A^n = PD^nP^{-1}}$

$$A^8 = PD^8P^{-1}$$

$$= \begin{bmatrix} -1 & +1 & 1 \\ -2 & 1 & 1 \\ -2 & +1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3^8 & 0 \\ 0 & 0 & 2^8 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 2 & 1 \\ 4 & -3 & -1 \end{bmatrix}$$

$$A^8 = \begin{bmatrix} -12098 & 12354 & -256 \\ -12098 & 12354 & -256 \\ 13122 & 13122 & 0 \end{bmatrix}$$

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$A^4 = PA^4P^{-1}$

$$\begin{bmatrix} -1 & 1 & 1 \\ -2 & 1 & 1 \\ -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 16 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ -2 & 2 & 1 \\ 4 & -3 & -1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -98 & 114 & -16 \\ -98 & 114 & -16 \\ -162 & 162 & 0 \end{bmatrix}$$

4. Show that $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ is a skew-Hermitian matrix and the unitary. Find the Eigen values and corresponding Eigen vectors of A.

Sol: If $A^{\theta} = -A$

$(\bar{A})^T = -A$ it is skew-Hermitian matrix

Given

$$A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$A = \begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{bmatrix}$$

$(\bar{A})^T =$

$$(\bar{A})^T = \begin{bmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{bmatrix}$$

$$A^\theta = -\begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix}$$

$$A^\theta = -A$$

Hence A is Skew-Hermitian matrix

For Unitary $A \cdot A^\theta = I$

$$= \begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix} \begin{bmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{bmatrix}$$

$$= \begin{bmatrix} -i^2 & 0 & 0 \\ 0 & -i^2 & 0 \\ 0 & 0 & -i^2 \end{bmatrix} \qquad \begin{bmatrix} \because \sqrt{-1} = i \\ -i^2 = 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

$$A = \begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix}, \quad (\bar{A})^T = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$$

$$\Rightarrow (\bar{A})^T (A) = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} \begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore\; A(\bar{A})^T = (\bar{A})^T A = I$

Hence A is unitary matrix

The characteristic equation of A is $|A-\lambda I| = 0$

i.e, $\begin{bmatrix} i-\lambda & 0 & 0 \\ 0 & 0-\lambda & i \\ 0 & i & 0-\lambda \end{bmatrix} = 0$ [Expand by R_1]

i.e, $(i-\lambda)(\lambda^2+1) = 0$

$\lambda^3 - i\lambda^2 + \lambda - i = 0$

$(\lambda+i)(\lambda-i)^2 = 0$

$\boxed{\therefore\; \lambda = -i, i, i}$

To find the eigen vectors for the corresponding eigen values, we will consider the matrix equation as

$(A-\lambda I)x = 0$

$$\begin{bmatrix} i-\lambda & 0 & 0 \\ 0 & 0-\lambda & i \\ 0 & i & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \text{— (1)}$$

eigen vector corresponding to $\lambda = -i$

Putting $\lambda = -i$ in equation (1) we get

$$\begin{bmatrix} 2i & 0 & 0 \\ 0 & i & i \\ 0 & i & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow\; 2ix_1 = 0,\quad x_2 + x_3 = 0,$

$\Rightarrow\; x_1 = 0,\quad x_2 = -x_3$

$\Rightarrow x_2 = 1 \quad , \quad x_3 = -1$

$\therefore$ eigen vector corresponding to $\lambda = -i$ is $X_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

eigen vector corresponding to $\lambda = i$

Putting $\lambda = i$ in (1), we get

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -i & i \\ 0 & i & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow -ix_2 + ix_3 = 0 \quad , \quad ix_2 - ix_3 = 0$

$\Rightarrow x_2 = x_3$

choose $x_1 = c_1$ where c_1 is arbitary. Then we have two linearly independent eigen vectors (with $x_1 = 0$, $x_2 = 1$ and $x_1 = 1$, $x_2 = 0$)

$\therefore$ eigen vectors corresponding to $\lambda = i$ are

$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

5. Find the diagonal matrix orthogonally similar to the following real symmetric matrix. Also obtain the transforming matrix. $A = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix}$

Sol:- The characteristic equation of A is

$$\begin{vmatrix} 7-\lambda & 4 & -4 \\ 4 & -8-\lambda & -1 \\ -4 & -1 & -8-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 7-\lambda & 4 & 0 \\ 4 & -8-\lambda & -9-\lambda \\ -4 & -1 & -9-\lambda \end{vmatrix} = 0 \text{ (Applying } C_3 + C_2)$$

$$\Rightarrow (-9-\lambda) \begin{vmatrix} 7-\lambda & 4 & 0 \\ 4 & -8-\lambda & 1 \\ -4 & -1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (-9-\lambda)\left[(7-\lambda)\{-8-\lambda+1\} - 4\{4+4\}\right] = 0$$

$$\Rightarrow (-9-\lambda)\left[(7-\lambda)(-\lambda-7) - 32\right] = 0$$

$$\Rightarrow (-9-\lambda)(\lambda^2 - 49 - 32) = 0.$$

$$\Rightarrow (-9-\lambda)(\lambda^2 - 81) = 0$$

$\therefore \lambda = 9, -9, -9$ are the eigen values.

Eigen vector corresponding to $\lambda=9$

It is given by $(A-\lambda I)X=0$

$$\Rightarrow \begin{bmatrix} 7-9 & 4 & -4 \\ 4 & -8-9 & -1 \\ -4 & -1 & -17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 4 & -4 \\ 4 & -17 & -1 \\ -4 & -1 & -17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying R_3+R_2, we get

$$\begin{bmatrix} -2 & 4 & -4 \\ 4 & -17 & -1 \\ 0 & -18 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -18x_2-18x_3=0 \Rightarrow x_3=-x_2$$

$$-2x_1+4x_2-4x_3=0 \Rightarrow -2x_1+4x_2+4x_2=0$$

$$\Rightarrow -2x_1+8x_2=0 \Rightarrow x_1=4x_2.$$

Take $x_2=k$. Then $x_1=4k$ and $x_3=-k$

Then $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4k \\ k \\ -k \end{bmatrix} = k\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$

$\therefore X_1 = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda=9$.

Eigen vector corresponding to $\lambda=-9$

It is given by $(A-\lambda I)X=0$

$$\Rightarrow \begin{bmatrix} 7+9 & 4 & -4 \\ 4 & -8+9 & -1 \\ -4 & -1 & -8+9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 16 & 4 & -4 \\ 4 & 1 & -1 \\ -4 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We have, $4x_1+x_2-x_3=0$. Take $x_2=k_1$, $x_3=k_2$.

Then $4x_1=x_3-x_2=k_2-k_1$

$$\therefore\ X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} \frac{k_2-k_1}{4} \\ k_1 \\ k_2 \end{bmatrix}=\frac{k_1}{4}\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}+\frac{k_2}{4}\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

$\therefore$ $\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$ are two mutually orthogonal vectors corresponding to $\lambda=-9$

Normalizing, we get.

$$P=\begin{bmatrix} 4/\sqrt{18} & -1/\sqrt{17} & 1/\sqrt{17} \\ 1/\sqrt{18} & 4/\sqrt{17} & 0 \\ -1/\sqrt{18} & 0 & 4/\sqrt{17} \end{bmatrix}$$

is the required orthogonal matrix that will diagonalise A.

Thus $P^{-1}AP=P^{T}AP=\text{diag}(9,-9,-9)$.

Q6. Verify Cayley-Hamilton theorem for the matrix

$$A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

Sol:- Given matrix: $A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

The characteristic equation for the given matrix is

$|A - \lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 8-\lambda & -8 & 2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1 \end{bmatrix} \right| = 0$$

$\Rightarrow 8-\lambda\left[(-3-\lambda)(1-\lambda)-8\right] - 4\left[-8(1-\lambda)+8\right] + 3\left[16 - 2(-3-\lambda)\right.$

$\Rightarrow (8-\lambda)\left[\lambda^2+2\lambda-11\right] - 4\left[8\lambda\right] + 3\left[2\lambda+22\right] = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 - \lambda - 22 = 0.$

→ Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation.

→ To verify Cayley-Hamilton theorem we have to prove that

$A^3 - 6A^2 - A + 22I = 0$

Now; $A^2 = A \cdot A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

$= \begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix}$

→ $A^3 = A^2 \cdot A = \begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix} \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

$= \begin{bmatrix} 214 & -296 & 206 \\ 88 & -115 & 70 \\ 69 & -100 & 69 \end{bmatrix}$

Now; $A^3 - 6A^2 - A + 22I$

$= \begin{bmatrix} 214 & -296 & 206 \\ 88 & -115 & 70 \\ 69 & -100 & 69 \end{bmatrix} - 6 \begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix} - \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} + 22 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow A^3 - 6A^2 - A + 22I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Hence Cayley-Hamilton theorem is verified.

Q7 Verify Cayley-Hamilton theorem $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

also find A^4 and A^{-1}.

S: Given matrix : $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$.

characteristic equation of A is given by $|A - \lambda I| = 0$

$\Rightarrow \begin{bmatrix} 1-\lambda & 2 & -1 \\ 2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{bmatrix} = 0$

$\Rightarrow (3-\lambda)\left[(1-\lambda)^2 - 4\right] - \lambda\left[-4 - 2(1-\lambda)\right] = 0$

$\Rightarrow \lambda^3 - 3\lambda^2 - 3\lambda + 9 = 0 \quad \text{—} \ (1)$

By Cayley-Hamilton theorem, matrix A should satisfy its characteristic equation.

i.e $A^2 - 3A^2 - 3A + 9I = 0$ — ②

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 24 & -21 \\ 6 & 21 & -24 \\ 6 & -6 & 3 \end{bmatrix}$$

$$A^3 - 3A^2 - 3A + 9I$$

$$= \begin{bmatrix} 3 & 24 & -21 \\ 6 & 21 & -24 \\ 6 & -6 & 3 \end{bmatrix} - 3\begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

Hence Cayley-Hamilton theorem is verified.

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→ To find A^{-1}

Multiplying eq ② with A^{-1} on both sides:

$$A^{-1}\left[A^3 - 3A^2 - 3A + 9I\right] = A^{-1}(0)$$

$$\Rightarrow A^2 - 3A - 3I + 9A^{-1} = 0$$

$$\Rightarrow 9A^{-1} = 3A + I - A^2$$

$$\Rightarrow A^{-1} = 1/9\,(3A + 3I - A^2)$$

$$= 1/9\left\{\begin{bmatrix} 3 & 6 & -3 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix}\right\}$$

$$= 1/9\begin{bmatrix} 3 & 0 & 3 \\ 6 & -3 & 0 \\ 6 & -6 & 3 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 1/3 \\ 2/3 & -1/3 & 0 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$\Rightarrow A\left[A^3 - 3A^2 - 3A + 9I\right] = 0$$

$$\Rightarrow A^4 - 3A^3 - 3A^2 + 9A = 0$$

$$\Rightarrow A^4 = 3A^3 + 3A^2 - 9A$$

$$= \begin{bmatrix} 9 & 72 & -63 \\ 18 & 63 & -72 \\ 18 & -18 & 9 \end{bmatrix} + \begin{bmatrix} 9 & 18 & -18 \\ 0 & 27 & -18 \\ 0 & 0 & 9 \end{bmatrix} - \begin{bmatrix} 9 & 18 & -9 \\ 18 & 9 & -18 \\ 18 & -18 & 9 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 9 & 72 & -72 \\ 0 & 81 & -72 \\ 0 & 0 & 9 \end{bmatrix}$$

Q8. Find the nature of quadratic form, index and signature of $10x^2+2y^2+5z^2-4xy-10xz+6yz$.

Sol: The given quadratic form is $10x^2+2y^2+5z^2-4xy-10xz+6yz$.

Its matrix is given by :- $A = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$

⇒ we write $A = I_3 A I_3$

i.e $\begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

⇒ $\begin{bmatrix} 10 & -2 & -5 \\ 0 & 8 & 10 \\ 0 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(By applying $R_2 \to R_1 + 5R_2$; $R_3 \to 2R_3 + R_1$)

⇒ $\begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 20 \\ 0 & 20 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$C_2 \to 5C_2 + C_1$; $C_3 \to 2C_3 + C_1$

$$\Rightarrow \begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 20 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & -5 & 4 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$R_3 \to 2R_3 - R_2$

$$\Rightarrow \begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 20 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & -5 & 4 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & -5 \\ 0 & 0 & 4 \end{bmatrix}$$

$C_3 \to 2C_3 - C_2$

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 20 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & -5 & 4 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & -5 \\ 0 & 0 & 4 \end{bmatrix}$$

Thus the quadratic form is reduced to normal form.

$D = P^T A P$

$$D = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & -5 \\ 0 & 0 & 4 \end{bmatrix}$$

Linear transformation $X = PY$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & -5 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$\Rightarrow x = y_1 + y_2 + y_3 \, ; \; y = 5y_2 - y_3 \, ; \; z = 4y_3$

The given quadratic form is reduced to

$X^T A X = (PY)^T A (PY) = Y^T (P^T A P) Y = Y^T D Y$

$$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 10y_1^2 + 40y_2^2$$

Nature of the quadratic form is positive semi-definite

Here $r = 2$, $n = 3$, $s = 2$

Index (s) = no. of positive terms in normal form $= 2$

Signature $= 2s - r = 2(2) - 2 = 2$

9) Reduce the quadratic form to the canonical form

$2x^2 + 5y^2 + 3z^2 + 4xy$

Sol. Given quadratic form is $2x^2 + 5y^2 + 3z^2 + 4xy$

Given quadratic form to matrix form is

$$\begin{bmatrix} x^2 & \frac{xy}{2} & \frac{xz}{2} \\ \frac{xy}{2} & y^2 & \frac{yz}{2} \\ \frac{xz}{2} & \frac{zy}{2} & z^2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

we write $A = I_3 A I_3$

we will Perform elementary operations on A in L.H.S. The corresponding row operations will be performed on pre factor of A and corresponding column operations will be Performed on post factor of A in R.H.S.

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$;

$C_2 \to C_2 - C_1$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \rightarrow \frac{R_1}{\sqrt{2}}$, $C_1 \rightarrow \frac{C_1}{\sqrt{2}}$, $R_2 \rightarrow \frac{R_2}{\sqrt{3}}$, $C_2 \rightarrow \frac{C_2}{\sqrt{3}}$, $R_3 \rightarrow \frac{R_3}{\sqrt{3}}$, $C_3 \rightarrow \frac{C_3}{\sqrt{3}}$

we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{3} & 1/\sqrt{3} & 0 \\ 0 & 0 & 1/\sqrt{3} \end{bmatrix} A \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & 1/\sqrt{3} \end{bmatrix}$$

This is of form $D = P^T A P$, where $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a diagonal matrix and $P^T = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{3} & 1/\sqrt{3} & 0 \\ 0 & 0 & 1/\sqrt{3} \end{bmatrix}$

The canonical form is $y_1^2 + y_2^2 + y_3^2$ which is given by $X = PY$ where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

10) Reduce the quadratic form $3x^2+5y^2+3z^2-2yz+2zx-2xy$ to canonical from by orthogonal reduction

Sol. Given equation is $3x^2+5y^2+3z^2-2yz+2zx-2xy$

Given quadratic form to matrix form is

$$\begin{bmatrix} x^2 & \frac{xy}{2} & \frac{xz}{2} \\ \frac{xy}{2} & y^2 & \frac{yz}{2} \\ \frac{xz}{2} & \frac{zy}{2} & z^2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

character equation can be written as $(A-\lambda I)=0$

$$\Rightarrow \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \lambda \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} = 0$$

$\Rightarrow (3-\lambda)[(5-\lambda)(3-\lambda)-1] + 1[-1(3-\lambda)+1] + 1[1-1(5-\lambda)] = 0$

$\Rightarrow (3-\lambda)[15-8\lambda+\lambda^2-1] + [-3+\lambda+1] + [1-5+\lambda] = 0$

$\Rightarrow (3-\lambda)[14-8\lambda+\lambda^2] + [-2+\lambda] + [-4+\lambda] = 0$

$\Rightarrow 42-24\lambda+3\lambda^2-14\lambda+8\lambda^2-\lambda^3-2+\lambda-4+\lambda = 0$

$\Rightarrow -\lambda^3+11\lambda^2-36\lambda+36 = 0$

$$\boxed{\lambda = 6, 3, 2}$$

Eigen values are 6, 3, 2

To find the eigen vector to corresponding the matrix of eigen value. So we will consider the matrix as

$(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Now let $\lambda = 2$

then

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow R_2 \longrightarrow R_2 + R_1$

$R_3 \longrightarrow R_3 - R_1$

$$\begin{bmatrix} -1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\rho(A) = 2, \quad n = 3$

Parameter $= n - r$
$= 1 \,(k)$

let $\boxed{z = k}$

$\Rightarrow -x + 2y - z = 0$

$\Rightarrow -x + 2k - k = 0$

$\Rightarrow \boxed{x = k}$

$\Rightarrow -y + z = 0$

$-y + k = 0$

$\boxed{y = k}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Now let $\boxed{\lambda = 6}$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_2 \to 3R_2 - R_1$

$R_3 \to 3R_3 + R_1$

$$\begin{bmatrix} -3 & -1 & 1 \\ 0 & -2 & -4 \\ 0 & -4 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} -3 & -1 & 1 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\rho(A) = 2, \quad n = 3$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\rho(A) = 2 \ , \ n = 3$$

Parameter $= n - r$
$= 1 \ (k)$

let $\boxed{z = k}$

$\Rightarrow \quad x - y + z = 0$

$\Rightarrow \quad 2y = 0$

$\boxed{y = 0}$, $\boxed{x = -k}$, $\boxed{z = k}$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Now let $\boxed{\lambda = 3}$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_2 \longleftrightarrow R_1$

$$\begin{bmatrix} -1 & 2 & -1 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_3 \longrightarrow R_3 + R_1$

$$\begin{bmatrix} -1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_3 \longrightarrow R_3 + R_2$

Parameter $= n - r$
$= 1\ (k)$

let $\boxed{z = k}$

$\Rightarrow -3x - y + z = 0$
$-3x + 2k + k = 0$
$-3x = -3k$
$\boxed{x = k}$

$\Rightarrow -2y - 4z = 0$
$-2y - 4k = 0$
$-2y = 4k$
$\boxed{y = -2k}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Model matrix $= [x_1 \ \ x_2 \ \ x_3]$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P = \left[\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}, \frac{x_3}{\|x_3\|}\right]$$

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

Now $D = P^T A P$

$$D = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Canonical form :- $Y^T D Y$

$$\Rightarrow [y_1\ y_2\ y_3] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\Rightarrow [y_1\ y_2\ y_3] \begin{bmatrix} 2y_1 \\ 3y_2 \\ 6y_3 \end{bmatrix}$$

$$\Rightarrow 2y_1^2 + 3y_2^2 + 6y_3^2$$

It is the required canonical form,,

11) Reduce the quadratic form $3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 - 2x_1x_3 - 2x_2x_3$ into sum of squares form by an orthogonal transformation and give the matrix of transformation

Sol: Given quadratic form $3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 - 2x_1x_3 - 2x_2x_3$

Given equation in to matrix form is

$$= \begin{bmatrix} x_1^2 & \frac{x_1x_2}{2} & \frac{x_1x_3}{2} \\ \frac{x_1x_2}{2} & x_2^2 & \frac{x_2x_3}{2} \\ \frac{x_1x_3}{2} & \frac{x_3x_2}{2} & x_3^2 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic equation of A is $(A-\lambda I)=0$

$$\Rightarrow \begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} = 0$$

$\Rightarrow (3-\lambda)\,[(3-\lambda)(3-\lambda)-1] - 1\,[(3-\lambda)+1] + 1\,[-1-(3-\lambda)] = 0$

$\Rightarrow \quad (\lambda-1)(\lambda-4)^2 = 0$

$\therefore \; \lambda = 1, 4, 4$

eigen values are $1, 4, 4$

To find the eigen vector to corresponding the matrix of eigen value. so we will consider the matrix as $(A-\lambda I)x = 0$

$$\Rightarrow \begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

let $\lambda = 1$, then

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to 2R_2 - R_1$

$R_3 \to 2R_3 - R_1$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\rho(A) = 2$, $n=3$

Parameter $= n - r$

$= 1 \ (k)$

let $\boxed{x_3 = k}$

$\Rightarrow 2x_1 + x_2 + x_3 = 0$

$2x_1 + k + k = 0$

$\boxed{x_1 = -k}$

$\Rightarrow 3x_2 - 3x_3 = 0$

$3x_2 - 3k = 0$

$\boxed{x_2 = k}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = X_1$$

Now let $\lambda = 4$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to R_2 + R_1$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\rho(A) = 1$, $n=3$

Parameter $= n - r$

$= 2 \ (k_1, k_2)$

let $\boxed{x_3 = k_1}$, $\boxed{x_2 = k_2}$

Hence $x_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

Model matrix is $[x_1 \; x_2 \; x_3]$

$$= \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P = \left[\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}, \frac{x_3}{\|x_3\|} \right]$$

$$= \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & -1/2 \\ 1/\sqrt{3} & 0 & -1 \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/2 \end{bmatrix}$$

Since P is orthogonal, we have $P^T = P^{-1}$

Thus $D = P^{-1}AP = P^TAP$

$$D = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/2 & -1 & 1/2 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & -1/2 \\ 1/\sqrt{3} & 0 & -1 \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/2 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

canonical form :- Y^TDY

$$= [y_1 \; y_2 \; y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= [y_1 \; y_2 \; y_3] \begin{bmatrix} y_1 \\ 4y_2 \\ 4y_3 \end{bmatrix}$$

$\Rightarrow -x_1 + x_2 + x_3 = 0$

$\Rightarrow -x_1 + k_2 + k_1 = 0$

$$\boxed{x_1 = k_1 + k_2}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ k_2 \\ k_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ are the eigen vectors corresponding to $\lambda = 4$

consider $a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ orthogonal to $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

i.e, $\begin{bmatrix} a+b \\ b \\ a \end{bmatrix}$ orthogonal to $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow a + b + a = 0$

$2a + b = 0$

$b = -2a$

Required vector is

$$a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 2a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a-2a \\ -2a \\ a \end{bmatrix} = \begin{bmatrix} -a \\ -2a \\ a \end{bmatrix} = a \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore$ The vector orthogonal to $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

$$= y_1^2 + 4y_2^2 + 4y_3^2$$

This is the required canonical form

The orthogonal transformation which reduces the quadratic form to canonical form is given by $X = PY$

i.e, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$\Rightarrow x_1 = \frac{-1}{\sqrt{3}} y_1 + \frac{2}{\sqrt{6}} y_3$; $x_2 = \frac{1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 + \frac{1}{\sqrt{6}} y_3$;

$x_3 = \frac{1}{\sqrt{3}} y_1 - \frac{1}{\sqrt{2}} y_2 + \frac{1}{\sqrt{6}} y_3$

Hence P is the matrix of transformation